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BIFURCATION AND STABILITY OF THE RELATIVE EQUILIBRIA OF A GYROSTAT SATELLITE*

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The bifurcation and stability of the relative equilibria of a gyrostat satellite in the case when the rotor axis does not lie in a principal plane of the central triaxial ellipsoid of inertia of the system are investigated. The results are represented as a bifurcation diagram, on which the distribution of the degree of instability of the relative equilibria obeys the usual laws of bifurcation theory, with the role of bifurcation parameter being played by the gyrostatic torque of the rotor.

1. In a central Newton force field, we will consider the motion of a rigid body rigidly attached to the axis of rotation of a statically and dynamically balanced rotor. We shall assume that the rotor rotates relative to the body at a constant angular velocity Ω, and the centre of mass C of the system moves along an unperturbed Keplerian circular orbit at an orbital angular velocity ω.

We introduce two Cartesian coordinate frames of reference: the orbital frame $Cxyz$, whose z axis is directed along the readius-vector of the satellite's centre of mass, the x axis along the tangent to the orbit in the direction of motion of the centre of mass, the y axis along the normal to the plane of the orbit and a frame $Cx_1x_2x_3$ rigidly attached to the body of the satellite, whose axes point along the principal central axes of inertia of the gyrostat.

The transformed potential energy of the gravitational forces and forces of inertia acting on the satellite in the orbital frame, in units of ω^2, is given by (see /1/)

$$W=\frac{1}{2}\sum_{j=1}^{3}(3A_j\gamma_j^2-A_j\beta_j^2-2k_j\beta_j)$$

Here $A_1\leqslant A_2\leqslant A_3$ are the principal central moments of inertia of the gyrostat satellite, $k_j=J\Omega\omega^{-1}e_j$ are the projections on the x_j axes $(j=1,2,3)$ of the gyrostatic torque vector of the gyrostat, in units of ω, J is the axial moment of inertia of the rotor, e_j are the cosines of the angles between the rotor axis and the x_j axes and γ_j and β_j are the projections on the x_j axes of the unit vectors γ and β along the radius-vector of the mass centre of the satellite and the normal to the orbital plane; with this notation.

$$\pi_\gamma=\gamma_1^2+\gamma_2^2+\gamma_3^2-1=0,\quad \pi_\beta=\beta_1^2+\beta_2^2+\beta_3^2-1=0 \tag{1.1}$$
$$\pi_{\gamma\beta}=\gamma_1\beta_1+\gamma_2\beta_2+\gamma_3\beta_3=0$$

The equations of relative equlibrium of the gyrostat satellite (relative to the orbital coordinate frame) may be written as

$$\frac{\partial W^*}{\partial\gamma_1}=3[(A_1-\sigma)\gamma_1+\lambda\beta_1]=0,\quad \frac{\partial W_*}{\partial\beta_1}=3\lambda\gamma_1+(\nu-A_1)\beta_1-k_1=0\ (1\,2\,3) \tag{1.2}$$

$$2W_*=2W+6\lambda\pi_{\gamma\beta}+\nu\pi_\beta-3\sigma\pi_\gamma$$

where λ,σ,ν are undetermined Lagrange multipliers. Eqs.(1.2) must be taken together with Eqs.(1.1); this gives a system of nine equations in the same number of unknowns $\lambda,\sigma,\nu,\gamma_j,\beta_j$.

We fix some $\lambda\neq 0,\sigma,\nu$ and solve Eqs.(1.2) for γ_j,β_j:

$$\gamma_1=\lambda k_1\Phi_1^{-1},\quad \beta_1=(\sigma-A_1)k_1\Phi_1^{-1}\ (1\ 2\ 3) \tag{1.3}$$
$$\Phi_1=3\lambda^2+(\sigma-A_1)(\nu-A_1)\ (1\ 2\ 3)$$

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Substitution of (1.3) into (1.1) yields a system of three linear equations in the unknown k_1^2, k_2^2, k_3^2, from which, assuming that

$$\lambda A \neq 0, \quad A = (A_1 - A_2)(A_2 - A_3)(A_3 - A_1)$$

we obtain

$$k_1^2 = \frac{(A_3 - A_2) L_1 \Phi_1^2}{\lambda^2 A}, \quad L_1 = \lambda^2 + (\sigma - A_2)(\sigma - A_3) \quad (1\,2\,3) \tag{1.4}$$

As a result we can write (1.3) in the form

$$\gamma_1^2 = \frac{(A_3 - A_2) L_1}{A}, \quad \beta_1^2 = \frac{(A_3 - A_2)(\sigma - A_1)^2 L_1}{\lambda^2 A} \; (1\,2\,3) \tag{1.5}$$

To get a geometrical representation of the relative equilibria (1.5), (1.4), we will consider the domain D defined in the parameter space λ, σ, ν by the inequalities $L_1 > 0$, $L_2 < 0$, $L_3 > 0$. The points of D are represented by real values of γ_j, β_j, k_j computed from formulae (1.5) and (1.4). The domain D is a cylindrical body, its profile is formed by three circles $L_j = 0$ $(j = 1, 2, 3)$, which are analogous to the familiar Mohr's circles of elasticity theory. It follows from (1.5) that the orientation of the gyrostat body at relative equilibrium is independent of ν. To each point on the profile of D there correspond eight equilibrium positions, represented by the γ_j, β_j values defined in (1.3) by the eight different combinations of signs of k_j $(j = 1, 2, 3)$. To symmetric points with respect to the plane $\lambda = 0$ there correspond dynamically equivalent equilibria of the satellite, which differ by a rotation of the satellite about the vector β through an angle 180°.

2. Let us assume from now on that

$$k_j = k e_j \; (j = 1, 2, 3), \quad e_1^2 + e_2^2 + e_3^2 = 1, \quad k = J\Omega\omega^{-1}$$

where $-\infty < k < \infty$ is a real parameter.

Consider the problem of the relative equilibrium of the gyrostat satellite in the direct formulation, when the numbers A_j, e_j $(j = 1, 2, 3)$ are assumed to be known and we have to find all relative equilibria and investigate their evolution, bifurcation and stability as k varies from $-\infty$ to ∞.

We shall assume that

$$(A_1 - A_2)(A_2 - A_3)(A_3 - A_1) e_1 e_2 e_3 \neq 0 \tag{2.1}$$

and the directions of the x_j axes are so chosen that $e_j > 0$ $(j = 1, 2, 3)$.

Let Γ denote the curve defined in the space of the variables $k, \lambda, \sigma, \nu, \gamma_j, \beta_j$ $(j = 1, 2, 3)$ by Eqs.(1.1) and (1.2) together with condition (2.1). Since Eqs.(1.1) and (1.2) are invariant to replacement of $k, \lambda, \sigma, \nu, \gamma_j, \beta_j$ by 1) $-k, -\lambda, \sigma, \nu, \gamma_j, -\beta_j$; 2) $k, -\lambda, \sigma, \nu, -\gamma_j, \beta_j$; 3) $-k, \lambda, \sigma, \nu, -\gamma_j, -\beta_j$, respectively, it follows that Γ is a union of the four branches $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ defined by the following equations:

$$\begin{aligned}
&\Gamma_1: \lambda = \lambda(k), \quad \sigma = \sigma(k), \quad \nu = \nu(k), \quad \gamma = \gamma(k), \quad \beta = \beta(k) \\
&\Gamma_2: \lambda = -\lambda(-k), \quad \sigma = \sigma(-k), \quad \nu = \nu(-k), \quad \gamma = \gamma(-k), \quad \beta = -\beta(-k) \\
&\Gamma_3: \lambda = -\lambda(k), \quad \sigma = \sigma(k), \quad \nu = \nu(k), \quad \gamma = -\gamma(k), \quad \beta = \beta(k) \\
&\Gamma_4: \lambda = \lambda(-k), \quad \sigma = \sigma(-k), \quad \nu = \nu(-k), \quad \gamma = -\gamma(-k), \quad \beta = -\beta(-k)
\end{aligned} \tag{2.2}$$

Let Γ^* and Γ^{**} denote the projections of Γ on the λ, σ, ν, k and λ, σ, ν subspaces, respectively, and Γ_j^* and Γ_j^{**} $(j = 1, \ldots, 4)$ the branches of Γ^* and Γ^{**} corresponding to the branches Γ_j of Γ. The branches Γ_j^{**} are defined in parametric form by the first three equations of (2.2). We shall use the term "representative points" for the points on the curves Γ_j^{**} whose coordinates λ, σ, ν for fixed k are defined by the first three equations of (2.2). It follows from (2.2) that the pairs Γ_1^{**} and Γ_3^{**}, Γ_2^{**} and Γ_4^{**}, are symmetrically placed with respect to the plane $\lambda = 0$. The pairs Γ_1^{**} and Γ_4^{**}, Γ_2^{**} and Γ_3^{**}, coincide identically, but the representative points move on them in opposite directions as k varies from $-\infty$ to ∞. If k is fixed, passage from $\Gamma_1^{**}, \Gamma_2^{**}$ to $\Gamma_4^{**}, \Gamma_3^{**}$ inverts the direction of the vectors γ and β. It will be shown in Sect.4 that each of the curves Γ_j^{**} consists of two branches $\Gamma_j^{**(1)}$ and $\Gamma_j^{**(2)}$. The branches of Γ^* are symmetrically placed with respect to the hyperplanes $\lambda = 0$ and $k = 0$.

Let us investigate the behaviour of the curves $\Gamma, \Gamma^*, \Gamma^{**}$ as $k \to \pm\infty$. Letting $k \to \pm\infty$ in (1.2), we obtain

$$\beta_j = \varkappa e_j, \quad \varkappa = \lim_{k\to\pm\infty} \frac{k}{\nu} = \pm 1, \quad \gamma_j = \frac{\varkappa\lambda e_j}{\sigma - A_j} \quad (j = 1, 2, 3) \tag{2.3}$$

Substituting β_j, γ_j from (2.3) into (1.1), we obtain equations for λ, σ:

$$\sum_{j=1}^{3} \frac{e_j^2}{\sigma - A_j} = 0, \quad \lambda^2 = \left(\sum_{j=1}^{3} \frac{e_j^2}{(\sigma - A_j)^2} \right)^{-1} \tag{2.4}$$

The first of these equations has two roots: $A_1 < \sigma_* < A_2 < \sigma^* < A_3$. Substituting these values of σ into the second equation of (2.4), we obtain two values $\lambda_*^2, \lambda^{*2}$. Thus Γ has four asymptotes, defined by Eqs.(2.3), to which we must add the equations $\sigma = \sigma_*, \lambda = \pm\lambda_*$ and $\sigma = \sigma^*, \lambda = \pm\lambda^*$. The curve Γ^{**} also has four asymptotes, defined by the equations $\sigma = \sigma_*$, $\lambda = \pm\lambda_*$ and $\sigma = \sigma^*, \lambda = \pm\lambda^*$.

3. We now consider the equations $\Phi_i = 0$ $(i = 1, 2, 3)$, where Φ_i are the functions defined in (1.3). The equations define three identical cones in λ, σ, ν space, whose apices lie on the same straight line, at points O_i with coordinates $\lambda = 0, \sigma = A_i, \nu = A_i$; their axes are parallel and lie in the plane $\lambda = 0$ at 45° angles to the ν and σ axes. The cones $\Phi_i = 0$ intersect the cylinders $L_i = 0$ $(i = 1, 2, 3)$ in curves G_i which project onto the plane $\lambda = 0$ as pieces of hyperbolae

$$G_1\colon \ \nu = A_1 + \frac{3(\sigma - A_2)(\sigma - A_3)}{\sigma - A_1} \quad (1\ 2\ 3)$$

The cones $\Phi_i = 0$ intersect the cylinders $L_j = 0$ $(i, j = 1, 2, 3;\ i \neq j)$ in ellipses E_i, E_i', which lie in parallel planes and project onto the plane $\lambda = 0$ as segments of parallel straight lines:

$$E_1 : \nu = 3\sigma + A_1 - 3A_2; \quad E_1' : \nu = 3\sigma + A_1 - 3A_3 \quad (1\ 2\ 3)$$

It follows from (1.4) that if condition (2.1) holds and $k \neq 0$, then Γ^{**} cannot intersect the surface of the cones $\Phi_i = 0$ $(i = 1, 2, 3)$, by which D is divided into fourteen domains $D_j^\pm$ $(j = 1, \ldots, 7)$. The domains D_j^+ and D_j^- are symmetric to one another with respect to the plane $\lambda = 0$, with $\lambda > 0$ for D_j^+ and $\lambda < 0$ for D_j^-. The domains $D_j^\pm$ are defined by the following inequalities:

$$D_1^\pm\colon \Phi_2 < 0,\ \Phi_3 < 0; \quad D_2^\pm\colon \Phi_2 > 0,\ \Phi_3 < 0; \quad D_3^\pm\colon \Phi_2 < 0,\ \Phi_3 > 0$$
$$D_4^\pm\colon \Phi_1 > 0,\ \Phi_2 > 0; \quad D_5^\pm\colon \Phi_1 > 0,\ \Phi_2 < 0; \quad D_6^\pm\colon \Phi_1 < 0,\ \Phi_2 < 0,\ D_7^\pm : \Phi_1 < 0,\ \Phi_2 > 0$$

While $D_3^\pm, D_4^\pm, D_5^\pm$ are bounded, $D_1^\pm, D_2^\pm, D_6^\pm, D_7^\pm$ are unbounded.

4. Eqs.(1.1), (1.2) have the following solutions at $k = 0$:

$$\sigma = A_1, \ \nu = A_3, \ \lambda = 0, \ k = 0, \ \gamma_1 = \gamma = \pm 1, \ \beta_3 = \beta = \pm 1, \quad \gamma_2 = \gamma_3 = \beta_1 = \beta_2 = 0 \quad (1\ 2\ 3) \tag{4.1}$$

$$\sigma = A_2, \ \nu = A_3, \ \lambda = 0, \ k = 0, \ \gamma_2 = \gamma = \pm 1, \ \beta_3 = \beta = \pm 1, \quad \gamma_3 = \gamma_1 = \beta_1 = \beta_2 = 0 \quad (1\ 2\ 3) \tag{4.2}$$

corresponding to which are 24 equilibrium positions of the satellite, in which the x_1, x_2, x_3 axes coincide in some way with the x, y, z axes. Formulae (4.1) define three groups P_1, P_2, P_3 of relative equilibria, the values of the variables for P_2 and P_3 being obtained from (4.1) by a cyclic permutation of the indices 1, 2, 3. In (4.1) $\gamma = \pm 1, \beta = \pm 1$ and any combination of signs is admissible; hence each of the groups P_1, P_2, P_3 contains four equlibria. Formulae (4.2) give three more, analogous groups Q_1, Q_2, Q_3 of equilibrium positions. Corresponding to the equilibrium groups P_i, Q_i $(i = 1, 2, 3)$ in D are points P_i°, Q_i° with coordinates

$$P_1^\circ\colon \sigma = A_1, \ \nu = A_3, \ \lambda = 0 \ (1\ 2\ 3); \quad Q_i^\circ\colon \sigma = A_2, \ \nu = A_3, \ \lambda = 0 \ (1\ 2\ 3)$$

situated on the boundaries of the domains $D_j^\pm$ $(j = 1, \ldots, 7)$.

If $|k|$ is small, one branch of Γ^{**} will correspond to each of the equilibria (4.1), (4.2). Let us denote these branches by $P_i(k, \gamma, \beta)$, $Q_i(k, \gamma, \beta)$ $(i = 1, 2, 3;\ \gamma = \pm 1,\ \beta = \pm 1)$; using Eqs.(1.1) and (1.2), we obtain the following parametric representations for them, valid for small $|k|$:

$$P_1(k,\gamma,\beta):\sigma=A_1+\frac{e_1^2k^2}{16(A_3-A_1)}+\ldots,\quad \nu=A_3+\beta e_3 k+\frac{3e_1^2k^2}{16(A_3-A_1)}+\ldots,\tag{4.3}$$

$$\lambda=\frac{1}{4}\gamma e_1 k-\frac{\gamma\beta e_3 e_1 k^2}{16(A_3-A_1)}+\ldots\quad (\gamma=\pm1,\beta=\pm1)\quad (1\,2\,3)$$

$$Q_1(k,\gamma,\beta):\sigma=A_2+\frac{e_2^2k^2}{16(A_3-A_2)}+\ldots,\quad \nu=A_3+\beta e_3 k+\frac{3e_2^2k^2}{16(A_3-A_2)}+\ldots,\tag{4.4}$$

$$\lambda=\frac{1}{4}\gamma e_2 k-\frac{\gamma\beta e_3 e_2 k^2}{16(A_3-A_2)}+\ldots\quad (\gamma=\pm1,\beta=\pm1)\quad (1\,2\,3)$$

The representative points on Γ^{**}, when $k=0$, occupy the positions P_i°, Q_i° $(i=1,2,3)$. Formulae (4.3), (4.4) enable us to determine which of the domains $D_j^\pm$ $(j=1,\ldots,7)$ will contain the representative points when $k>0$ and $k<0$ (Table).

$P_i(k,\gamma,\beta)$	$k>0$	$k<0$	$Q_i(k,\gamma,\beta)$	$k>0$	$k<0$
$P_1(k,1,1)$	D_1^+	D_3^-	$Q_1(k,1,1)$	D_2^+	D_4^-
$P_1(k,1,-1)$	D_3^+	D_1^-	$Q_1(k,1,-1)$	D_4^+	D_2^-
$P_1(k,-1,1)$	D_1^-	D_3^+	$Q_1(k,-1,1)$	D_2^-	D_4^+
$P_1(k,-1,-1)$	D_3^-	D_1^+	$Q_1(k,-1,-1)$	D_4^-	D_2^+
$P_2(k,1,1)$	D_4^+	D_7^-	$Q_2(k,1,1)$	D_5^+	D_6^-
$P_2(k,1,-1)$	D_7^+	D_4^-	$Q_2(k,1,-1)$	D_6^+	D_5^-
$P_2(k,-1,1)$	D_4^-	D_7^+	$Q_2(k,-1,1)$	D_5^-	D_6^+
$P_2(k,-1,-1)$	D_7^-	D_4^+	$Q_2(k,-1,-1)$	D_6^-	D_5^+
$P_3(k,1,1)$	D_4^+	D_5^-	$Q_3(k,1,1)$	D_3^+	D_4^-
$P_3(k,1,-1)$	D_5^+	D_4^-	$Q_3(k,1,-1)$	D_4^+	D_3^-
$P_3(k,-1,1)$	D_4^-	D_5^+	$Q_3(k,-1,1)$	D_3^-	D_4^+
$P_3(k,-1,-1)$	D_5^-	D_4^+	$Q_3(k,-1,-1)$	D_4^-	D_3^+

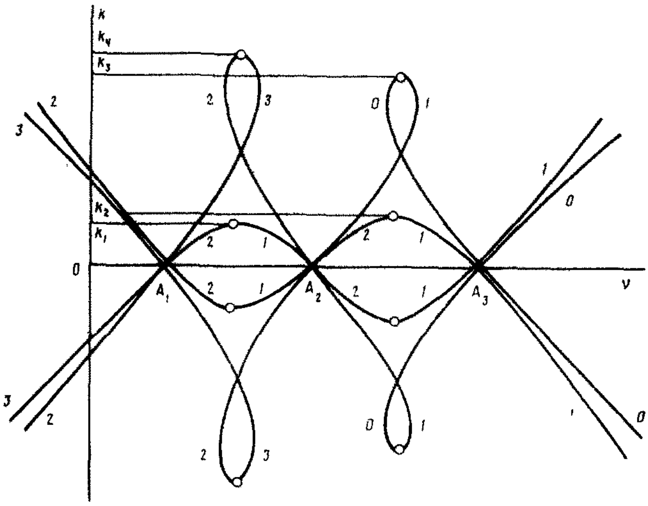
We now put the branches $P_i(k,\gamma,\beta)$, $Q_i(k,\gamma,\beta)$ $(i=1,2,3)$ together to form the following curves $\Gamma_l^{**(\varkappa)}$ $(l=1,\ldots,4;\ \varkappa=1,2)$:

$\Gamma_1^{**(1)}$: $P_1(k,1,1)$, $Q_3(k,1,-1)$, $P_2(k,1,1)$
$\Gamma_2^{**(1)}$: $P_1(k,-1,-1)$, $Q_3(k,-1,1)$, $P_2(k,-1,-1)$
$\Gamma_3^{**(1)}$: $P_1(k,-1,1)$, $Q_3(k,-1,-1)$, $P_2(k,-1,1)$
$\Gamma_4^{**(1)}$: $P_1(k,1,-1)$, $Q_3(k,1,1)$, $P_2(k,1,-1)$
$\Gamma_1^{**(2)}$: $Q_1(k,1,1)$, $P_3(k,1,-1)$, $Q_2(k,1,1)$
$\Gamma_2^{**(2)}$: $Q_1(k,-1,-1)$, $P_3(k,-1,1)$, $Q_2(k,-1,-1)$
$\Gamma_3^{**(2)}$: $Q_1(k,-1,1)$, $P_3(k,-1,-1)$, $Q_2(k,-1,1)$
$\Gamma_4^{**(2)}$: $Q_1(k,1,-1)$, $P_3(k,1,1)$, $Q_2(k,1,-1)$

To explain these formulae, let us describe, say, the structure and position of the curve $\Gamma_1^{**(1)}$ in D. We begin with the part $P_1(k,1,1)$ of the curve. If $k=0$ the representative point is in position P_1°; if $k>0$ it lies in D_1^+; as $k\to\infty$ it asymptotically approaches the asymptote $\sigma=\sigma_*$, $\lambda=\lambda_*$. If $k<0$ the representative point of $P_1(k,1,1)$ lies in D_3^- and at some $k=-k_3<0$ the curve $P_1(k,1,1)$ joins $Q_3(k,1,-1)$, on which the representative point lies, if $k=0$, at Q_3°; if $k<0$ it enters D_3^- and at $k=-k_3$ the curve $Q_3(k,1,-1)$ joins $P_1(k,1,1)$. If $k>0$ the representative point of $Q_3(k,1,-1)$ enters D_4^+ and at some $k=k_4>0$ $Q_3(k,1,-1)$ joins $P_2(k,1,1)$, on which the representative point, if $k=0$, occupies the position P_2°; if $k>0$ it enters D_4^+ and at $k=k_4$ $P_2(k,1,1)$ joins $Q_3(k,1,-1)$. If $k<0$ the representative point of $P_2(k,1,1)$ lies in D_7^- and as $k\to-\infty$ asymptotically approaches the asymptote $\sigma=\sigma_*$, $\lambda=-\lambda_*$. The structure of the other curves $\Gamma_l^{**(\varkappa)}$ is analogous.

The pairs of curves $\Gamma_1^{**(\varkappa)}$ and $\Gamma_3^{**(\varkappa)}$, $\Gamma_2^{**(\varkappa)}$ and $\Gamma_4^{**(\varkappa)}$ are symmetrically placed relative to the plane $\lambda=0$. The pairs $\Gamma_1^{**(\varkappa)}$ and $\Gamma_4^{**(\varkappa)}$, $\Gamma_2^{**(\varkappa)}$ and $\Gamma_3^{**(\varkappa)}$ coincide, but their representative points move in opposite directions as k varies from $-\infty$ to ∞. The curves $\Gamma_l^{**(1)}$ $(l=1,\ldots,4)$ lie in the part of D for which $\sigma<A_2$, whereas $\Gamma_l^{**(2)}$ lie in the part for which $\sigma>A_2$.

In λ,σ,ν,k space the branches $\Gamma_l^{**(\varkappa)}$ $(l=1,\ldots,4;\ \varkappa=1,2)$ of Γ^{**} correspond to the branches $\Gamma_l^{*(\varkappa)}$ of Γ^*, whose projection on the k,ν plane is shown in the figure. The curves are actually double (they consist of two "banks"). To different banks there correspond relative equilibria in which the vector $\boldsymbol{\gamma}$ has opposite directions. Hence we conclude that there are eight bifurcation values $k=\pm k_j$ $(j=1,\ldots,4)$ of the parameter k; when the parameter goes through these values, the number of relative equilibria changes by four, the maximum number being twenty-four (if $|k| < k_1$) and the minimum eight (if $|k| > k_4$). The digits 0, 1, 2, 3 on the branches of the curves indicate the degree of instability of the appropriate equilibrium; this degree of instability changes only at bifurcation points, corresponding to the summits of "humps" and the bases of "hollows".



5. The sufficient conditions for the relative equilibria of the gyrostat satellite (1.5), (1.4) to be stable /1/ may be written in terms of the parameters λ, σ, ν as follows /2/:

$$a > 0,\ 2a\nu + b > 0,\ \Delta = a\nu^2 + b\nu + c > 0 \tag{5.1}$$
$$a = \lambda^{-2}H,\ b = 3H' - 2\sigma\lambda^{-2}H - \lambda^{-4}H^2$$
$$c = {}^9/_2\lambda^2H'' + 6H - 3\sigma H' + (\sigma^2 - 3H')\lambda^{-2}H + \sigma\lambda^{-4}H^2$$
$$H = (\sigma - A_1)(\sigma - A_2)(\sigma - A_3),\ H' = dH/d\sigma$$

Consider the following two surfaces in λ, σ, ν space:

$$\nu = \nu^{\pm}(\lambda, \sigma),\ \nu^{\pm} = (b \pm \sqrt{b^2 - 4ac})/(2a)$$

defined by the equation $\Delta = 0$. The functions $\nu = \nu^{\pm}$ take real values for all admissible values of $\lambda \neq 0$. The surface $\nu = \nu^+$ intersects the cylinders $L_i = 0\ (i = 1, 2, 3)$ in curves G_i. The surface $\nu = \nu^-$ intersects the cylinders $L_i = 0$ in ellipses E_i'' which lie in parallel planes and project onto the plane $\lambda = 0$ as segments of parallel straight lines

$$E_1'' : \nu = 7\sigma - 3(A_2 + A_3)\ (1\ 2\ 3)$$

The surface $\nu = \nu^+$ has a discontinuity at $\sigma = A_2$. As $\sigma \to A_2$ it asymptotically approaches the plane $\sigma = A_2$.

Conditions (5.1) are equivalent to the conditions /2/

$$a > 0,\ \nu > \nu_2,\ \nu_1 = \min(\nu^+, \nu^-),\ \nu_2 = \max(\nu^+, \nu^-) \tag{5.2}$$

It follows from (5.2) that the degree of instability χ of equilibria with $\nu > \nu_2$, $\nu_1 < \nu < \nu_2$, $\nu < \nu_1$ is 0, 1, 2 if $a > 0$ and 1, 2, 3 if $a < 0$.

It was shown above that the curves $\Gamma_l^{**(1)}\ (l = 1, \ldots, 4)$ are situated in the part of D for which $\sigma \leqslant A_2$, and the curves $\Gamma_l^{**(2)}$, in the part for which $\sigma \geqslant A_2$. For the former, therefore, $a > 0$, and for the latter, $a < 0$. Consequently, the degrees of instability on the curves $\Gamma_l^{**(1)}$ are $\chi = 0, 1, 2$, while on $\Gamma_l^{**(2)}$ we have $\chi = 1, 2, 3$.

The results obtained by analysing the stability conditions (5.2) for the equilibria (1.5), (1.4) are shown in the figure, where the digits 0, 1, 2, 3 labelling the branches of the curves indicate the degree of instability of the appropriate equilibrium; this degree of instability changes only at bifurcation points, corresponding to the summits of "humps" and the bases of "hollows".

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STABILITY OF A SOLID CONTAINING A FLUID MOVING IN A FLUID*

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A solid, suspended on a horizontal rod, with three pairwise orthogonal axes of symmetry which is placed in an ideal incompressible fluid executing a vortex-free motion is considered. The body has a cavity containing a fluid which is covered by an elastic membrane. Under certain conditions, the equations of motion of the system permit uniform translational motions of the whole system as a single body. The stability conditions for such motions are given.

1. Formulation of the problem. Let a solid S with three pairwise orthogonal axes of symmetry move in an ideal incompressible fluid of density ρ which is at rest at infinity. The body has a cavity containing an ideal fluid of density ρ' covered by an elastic membrane Σ of density ρ'', the contour of which, $\partial\Sigma$, is fixed onto the wall of the cavity. The "external fluid - body - internal fluid - membrane" system is located in a uniform gravitational field with an acceleration g.

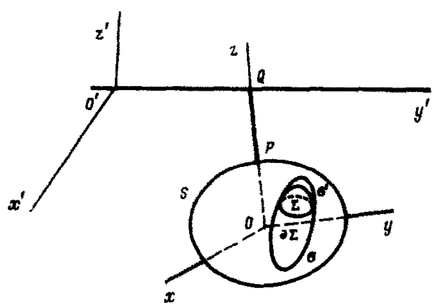

Fig.1

Let us now introduce three orthogonal coordinate systems: the inertial coordinate system $O'x'y'z'$ with the unit vectors $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ and with the z'-axis directed along the ascending vertical, a moving $Oxyz$ coordinate system with the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the axes of which coincide with the axes of symmetry of the body S, and the coordinate system ΩXYZ, the axes of which are parallel to the x-, y- and z-axes and the ΩXY plane contains the area Σ which is occupied by the membrane in the undeformed state. We shall assume that the body is suspended from a horizontal bar directed along the y'-axis using a solid rod PQ of negligibly small mass located along the z-axis and that $OP = a$ and $PQ = L$. We shall neglect the friction and action of the external fluid on the rod when the end of this rod Q moves along the axis of suspension (see Fig.1).

Let τ be the part of the cavity which is occupied by the fluid and let σ be the part of its wall which is wetted by the fluid. We will assume that the membrane is constantly in contact with the fluid and that the part of the cavity which is enclosed between the membrane

*Prikl.Matem.Mekhan., 55,4,572-577,1991